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On the Numerical Calculation of Elliptic Integrals of the
first and second kind and the Elliptic Functions of Jacobi

by

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Introduction

In this paper formulae will be given, which can be used for accurate and fast computation of Elliptic Integrals and Jacobian Elliptic Functions.

Extensive theory can be found in a monograph by A.V. King [1]. All formulae are based upon Gauss theory of the arithmetico-geometrical means [2], and Legendre's method of computation. Programs to obtain 12 significant figures were set up in ALGOL-60, and run on the X1 computer of the Mathematical Centre. Recently a series expansion method has been proposed by DiDonati and Hershey [3], which they claim to be superior to Legendre's method. Their ALGOL program [4] was also run on the X1.

A judicious application of Gauss' theory, however resulted in a more compact and faster program than the series expansion program.

We have the usual definitions for the Incomplete Elliptic integrals

$$(1.1) \quad F(k, \varphi) = \int_0^{\varphi} (1 - k^2 \sin^2 \varphi)^{-\frac{1}{2}} d\varphi \quad \text{we assume } 0 \leq k \leq 1,$$

$$(1.2) \quad E(k, \varphi) = \int_0^{\varphi} (1 - k^2 \sin^2 \varphi)^{+\frac{1}{2}} d\varphi.$$

Moreover we consider the function¹⁾

$$(1.3) \quad B(k, \varphi) = \int_0^{\varphi} \cos^2 \varphi (1 - k^2 \sin^2 \varphi)^{-\frac{1}{2}} d\varphi, \text{ which is}$$

related to F and E by

$$(1.4) \quad B(k, \varphi) = \frac{1}{k^2} [E(k, \varphi) - k'^2 F(k, \varphi)], \quad k' = (1 - k^2)^{\frac{1}{2}}.$$

For small values of k both numerator and denominator of (1.4) become small, which results in a loss of significant figures if B is calculated from tables of E and F.

¹⁾ In the Technical note TN 28, Mathematical Centre, November 1962, a table in 11 decimals is given of $B(k, \varphi)$ with $\varphi = 1^\circ(1^\circ)90^\circ$, $k = \sin [1^\circ(1^\circ)45^\circ]$.

The Complete Elliptic Integrals are defined by

$$(1.5) \quad K(k) = F(k, \frac{\pi}{2}) ,$$

$$(1.6) \quad \mathcal{E}(k) = E(k, \frac{\pi}{2}) ,$$

$$(1.7) \quad B(k) = B(k, \frac{\pi}{2}) .$$

If $F(k, \varphi) = u$, the inverse function is called the amplitude function $\varphi = \text{am}(u, k)$.

The Jacobian Elliptic functions are defined by

$$(1.8) \quad \text{sn}(u, k) = \sin \text{am}(u, k) ,$$

$$(1.9) \quad \text{cn}(u, k) = \cos \text{am}(u, k) ,$$

$$(1.10) \quad \text{dn}(u, k) = (1 - k^2 \text{sn}^2(u, k))^{\frac{1}{2}} .$$

Arithmetico-geometrical means

Let a_0 and b_0 be two positive numbers. We then define the following sequences: $\{a_n\}$, $\{b_n\}$ as follows

$$a_{n+1} = \frac{a_n + b_n}{2} ; b_{n+1} = (a_n b_n)^{\frac{1}{2}} ; n=0, 1, 2, \dots .$$

It can easily be shown that these sequences converge quadratically to the same limit, which is denoted by $\text{ag } M(a_0, b_0)$.

In the following the letters a_n and b_n will be used exclusively if they occur in the just mentioned sequences.

The Incomplete Elliptic Integrals

Let us define the following functions

$$(2.1) \quad F^*(a, b; \varphi) = \frac{1}{a} F\left(\left(1 - \frac{b^2}{a^2}\right)^{\frac{1}{2}}, \varphi\right) = \int_0^\varphi (a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^{-\frac{1}{2}} d\varphi ,$$

$$(2.2) \quad E^*(a, b; \varphi) = a E\left(\left(1 - \frac{b^2}{a^2}\right)^{\frac{1}{2}}, \varphi\right) = \int_0^\varphi (a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^{\frac{1}{2}} d\varphi,$$

$$(2.3) \quad F'(a, b; \varphi) = \frac{1}{a} F\left(\frac{b}{a}, \varphi\right) = \int_0^\varphi (a^2 - b^2 \sin^2 \varphi)^{-\frac{1}{2}} d\varphi,$$

$$(2.4) \quad E'(a, b; \varphi) = a E\left(\frac{b}{a}, \varphi\right) = \int_0^\varphi (a^2 - b^2 \sin^2 \varphi)^{\frac{1}{2}} d\varphi.$$

The Landen transformation [5] is given by

$$(2.5) \quad \tan \varphi_1 = \frac{(1+k') \tan \varphi}{1-k' \tan^2 \varphi},$$

or if we write $k' = b_0/a_0$, $0 < a \leq 1$, $0 \leq b \leq 1$, by

$$(2.6) \quad \tan \varphi_1 = \frac{(a_0 + b_0) \tan \varphi}{a_0 - b_0 \tan^2 \varphi}.$$

Substitution in (2.1) and (2.2) gives

$$(2.7) \quad F^*(a_0, b_0; \varphi) = \frac{1}{2} F^*(a_1, b_1; \varphi_1) \text{ where } a_1 = \frac{a_0 + b_0}{2}; \quad b_1 = (a_0 - b_0)^{\frac{1}{2}},$$

$$(2.8) \quad E^*(a_0, b_0; \varphi) = E^*(a_1, b_1; \varphi_1) - \frac{b_0 a_0}{2} F^*(a_1, b_1; \varphi_1) + \frac{a_0 - b_0}{2} \sin \varphi_1.$$

If we transform in the opposite direction writing

$$\tan \varphi = \frac{(1+k'_1) \tan \varphi_1}{1-k'_1 \tan^2 \varphi_1} \text{ where } k_1 = \frac{2\sqrt{k}}{1+k},$$

we get with $k = b_0/a_0$

$$(2.9) \quad a_1 \cot \varphi_1 = \frac{1}{2} \left\{ a_0 \cot \varphi + (a_0^2 \cot^2 \varphi + a_0^2 - b_0^2)^{\frac{1}{2}} \right\}.$$

Substitution in (2.3) and (2.4) gives

$$(2.10) \quad F'(a_0, b_0; \varphi) = F'(a_1, b_1; \varphi_1) \text{ where } a_1 = \frac{a_0 + b_0}{2}; \quad b_1 = (a_0 - b_0)^{\frac{1}{2}},$$

$$(2.11) \quad E'(a_0, b_0; \varphi) = 2 E'(a_1, b_1; \varphi_1) + (a_0 - b_0) a_1 F'(a_1, b_1; \varphi_1) - b_0 \sin \varphi.$$

Gauss' transformation [2] is given by

$$\sin \varphi_1 = \frac{(1+k)\sin \varphi}{1+k \sin^2 \varphi} \quad \text{which becomes if we substitute } k=b_0/a_0$$

$$(2.12) \quad \sin \varphi_1 = \frac{(a_0+b_0)\sin \varphi}{a_0+b_0 \sin^2 \varphi}.$$

Substitution in (2.3) and (2.4) gives

$$(2.13) \quad F'(a_0, b_0; \varphi) = \frac{1}{2} F'(a_1, b_1; \varphi_1) \quad \text{with } a_1 = \frac{a_0+b_0}{2}; \quad b_1 = (a_0 - b_0)^{\frac{1}{2}},$$

$$(2.14) \quad E'(a_0, b_0; \varphi) = E'(a_1, b_1; \varphi_1) + \frac{a_0^2 - b_0^2}{4} F'(a_1, b_1; \varphi_1) - b_0 \sin \varphi \cos \varphi_1.$$

An iterative program based on (2.14) may lead to the loss of significant figures, as $\cos \varphi_1$ must be calculated from $\sin \varphi_1$. Reversion of this transformation results in

$$\sin \varphi = \frac{(1+k_1)\sin \varphi_1}{1+k_1 \sin^2 \varphi_1} \quad \text{with } k_1 = \frac{1-k'}{1+k'}, \quad \text{or if we write } k' = b_0/a_0$$

$$\sin \varphi = \frac{2a_0 \sin \varphi_1}{(a_0+b_0) + (a_0-b_0)\sin^2 \varphi_1}$$

Substitution in (2.1) and (2.2) gives

$$(2.15) \quad F^*(a_0, b_0; \varphi) = F^*(a_1, b_1; \varphi_1),$$

$$(2.16) \quad E^*(a_0, b_0; \varphi) = 2 E^*(a_1, b_1; \varphi_1) - b_1 a_1 F^*(a_1, b_1; \varphi_1) + (a_0 - b_0) \sin \varphi_1 \cos \varphi.$$

It leads however to the same difficulty.

We therefore based our programs on iterative use of formulae (2.7), (2.8), (2.10) and (2.11).

One can easily verify that the calculation of $ag M(1, b)$, where $1 \geq b > 0.3$ needs at most three cycles, and for $b > 0.9539 = (1 - 0.3^2)^{\frac{1}{2}}$ only two cycles, to obtain 12 significant figures.

Thus when $k < 0.9539$ we choose $a_0 = 1$, $b_0 = k'$ and base our program on formulae (2.7) and (2.8).

$$\text{With } \tan \varphi_1 = \frac{(a_0 + b_0) \tan \varphi}{a_0 - b_0 \tan^2 \varphi} \text{ or } a_1 \cot \varphi_1 = \frac{1}{2} (a_0 \cot \varphi - \frac{b_0 a_0}{a_0 \cot \varphi})$$

and $a_0 = 1$, $b_0 = k'$, we get, after rearranging terms, so that loss of significant figures can not occur,

$$(2.17) \quad F(k, \varphi) = F^*(1, k'; \varphi) \approx \frac{\varphi_4}{8(a_3 + b_3)},$$

$$(2.18) \quad E(k, \varphi) = E^*(1, k'; \varphi) \approx$$

$$\frac{1}{4} \left\{ \frac{\varphi_4}{a_3 + b_3} \left[\frac{1}{4} - \sum_{i=0}^3 2^{i-3} (a_i^2 - b_i^2) \right] + \sum_{i=0}^3 (a_i^2 - b_i^2) \frac{\sin \varphi_{i+1}}{a_{i+1}} \right\},$$

$$(2.19) \quad B(k, \varphi) \approx \left\{ \frac{1}{2} - \sum_{i=1}^3 2^{i-1} p_i \right\} \frac{\varphi_4}{8(a_3 + b_3)} + \sum_{i=0}^3 \frac{p_i}{4} \frac{\sin \varphi_{i+1}}{a_{i+1}},$$

where
$$p_i = \frac{a_{i-1} - b_{i-1}}{4(a_{i-1} + b_{i-1})} \cdot p_{i-1}, \quad p_0 = 1.$$

φ_4 is calculated with the formula

$$\varphi_4 = \arctan \left(\frac{1}{\cot \varphi_4} \right) + \left\{ \sum_{j=0}^4 2^{4-j} \frac{1 - \operatorname{sgn}(\cot \varphi_j)}{2} + \frac{1 - \operatorname{sgn}(\cot \varphi_4)}{2} \right\} \cdot \frac{\pi}{2}.$$

When $k \geq 0.9539$ we choose $a_0 = 1$, $b_0 = k$ and base our program on formulae (2.10) and (2.11).

Iterating two times, we get

$$(2.20) \quad F(k, \varphi) = F'(1, k; \varphi) \approx \frac{2}{a_2 + b_2} \ln \frac{1 + (1 + \cot^2 \varphi_3)^{\frac{1}{2}}}{\cot \varphi_3},$$

$$(2.21) \quad E(k, \varphi) = E'(1, k; \varphi) \approx \frac{1}{a_2 + b_2} \left\{ k'^2 + \sum_{i=1}^2 2^i (a_i^2 - b_i^2) \right\} \cdot \\ \cdot \ln \frac{1 + (1 + \cot^2 \varphi_3)^{\frac{1}{2}}}{\cot \varphi_3} + \sum_{i=0}^2 2^i b_i (\sin \varphi_3 - \sin \varphi_i) + \sin \varphi_3,$$

$$(2.22) \quad B(k, \varphi) \approx \frac{1}{k^2} \left[\frac{1}{a_2 + b_2} \left\{ -k'^2 + \sum_{i=1}^2 2^i (a_i^2 - b_i^2) \right\} \ln \frac{1 + (1 + \cot^2 \varphi_3)^{\frac{1}{2}}}{\cot \varphi_3} + \right. \\ \left. + \sum_{i=0}^2 2^i b_i (\sin \varphi_3 - \sin \varphi_i) + \sin \varphi_3 \right].$$

A fast and compact ALGOL-60 program was easily obtained, and run on the X1. Comparing tested values with a 12 decimal table of Legendre - Emde [6], we found a relative error of the order 10^{-12} .

The ALGOL program of the series expansion method by DiDonati and Hershey ([3], [4]) was also run on the X1.

It turned out that our program is about four times faster.

Further on we observed that for $\varphi = 1^\circ$ and $k = \sin 89^\circ$, the series expansion program did not give an answer within an acceptable time, in fact the calculation needed more than 10 minutes.

The Complete Elliptic Integrals

We only write down the formulae, which can be easily derived from the foregoing formulae.

We mention here that Morgan Ward [7] derived similar formulae for the Complete Elliptic Integral of the third kind.

When $k < 0.9539$ we get

$$K(k) \approx \frac{\pi}{a_3 + b_3} \quad \text{where } a_0 = 1; b_0 = k',$$

$$\xi(k) \approx \left\{ \frac{a_0^2 + b_0^2}{2} - \sum_{i=1}^3 2^{i-1} (a_i^2 - b_i^2) \right\} \frac{\pi}{a_3 + b_3},$$

$$B(k) \approx \left\{ \frac{1}{2} - \sum_{i=1}^3 2^{i-1} p_i \right\} \frac{\pi}{a_3 + b_3} \quad \text{where } p_1 = \frac{a_{1-1} - b_{1-1}}{4(a_{1-1} + b_{1-1})} \cdot p_{1-1}, p_0 = 1.$$

When $k \geq 0.9539$ we get if we use

$$K(k) = \sum_{n=0}^{\infty} \left\{ \frac{(\frac{1}{2})_n}{n!} \right\}^2 \left\{ \psi(n+1) - \psi(n+\frac{1}{2}) - \ln k' \right\} k'^{2n},$$

$$K(k) \approx \frac{1}{4a_3} \ln \frac{a_3 a_2 a_1^2}{(\frac{k'}{4})^4},$$

$$\xi(k) \approx a_3 + \sum_{i=0}^2 2^i (a_i^2 - b_i^2) \frac{1}{8a_3} \ln \frac{a_3 a_2 a_1^2}{(\frac{k'}{4})^4},$$

$$B(k) \approx \frac{1}{k^2} \left\{ a_3 + \left[-(a_0^2 - b_0^2) + 2(a_1^2 - b_1^2) + 4(a_2^2 - b_2^2) \right] \frac{1}{8a_3} \ln \frac{a_3 a_2 a_1^2}{(\frac{k'}{4})^4} \right\}.$$

Results were checked with a 15 decimal table [8].

The relative errors were all of the order 10^{-12} .

The Elliptic Functions of Jacobi

Inverting formula (2.15) by which we can obtain $F(k, \varphi)$, we get $\sin \varphi_0$ with known $F(k, \varphi)$.

$$\text{As } \frac{\varphi_4}{a_4} \approx u = F(k, \varphi) \text{ we find } \sin \varphi_0 \text{ by } \sin \varphi_1 = \frac{2a_1 \sin \varphi_{1+1}}{(a_1 + b_1) + (a_1 - b_1) \sin^2 \varphi_{1+1}},$$

with $a_0 = 1, b_0 = k'$ for $k < 0.9539$.

(the same is done by Salzer [9])

When $\varphi_4 \leq \frac{\pi}{4}$ then $\varphi_0 \leq \frac{\pi}{4}$, we use

$$\text{cn}(u, k) \approx (1 - \sin^2 \varphi_0)^{\frac{1}{2}} \text{ (which is not suitable when } \varphi_0 > \frac{\pi}{4}!).$$

When $\varphi_4 > \frac{\pi}{4}$ we use the well-known relation

$$(3.1) \quad \text{cn}(u, k) = k' \frac{\text{sn}(K-u, k)}{\text{dn}(K-u, k)} \approx \frac{k' \sin \varphi_0^*}{(1 - k'^2 \sin^2 \varphi_0^*)^{\frac{1}{2}}} \text{ where } K=K(u),$$

and $\sin \varphi_0^*$ is obtained from $\varphi_4^* = a_4(K-u) = \frac{\pi}{2} - a_4 u$.

For $\text{dn}(u, k)$ we use similar formulae

$$\text{dn}(u, k) \approx (1 - k^2 \sin^2 \varphi_0)^{\frac{1}{2}} \text{ for } \varphi_4 \leq \frac{\pi}{4},$$

$$(3.2) \quad \text{dn}(u, k) = \frac{k'}{\text{dn}(K-u, k)} \approx \frac{k'}{(1 - k'^2 \sin^2 \varphi_0^*)^{\frac{1}{2}}} \text{ for } \varphi_4 > \frac{\pi}{4}.$$

When $k > 0.9539$ we invert formula (2.10) and obtain $\tan \varphi_3 = \sinh(a_3 u)$, then $\tan \varphi_0$ can be calculated with

$$\tan \varphi_i = \frac{2 a_i \tan \varphi_{i+1}}{(a_i + b_i) - (a_i - b_i) \tan^2 \varphi_{i+1}}.$$

$$\text{Then } \text{sn}(u, k) = \sin \varphi_0 = \frac{\tan \varphi_0}{(1 + \tan^2 \varphi_0)^{\frac{1}{2}}}, \quad \text{cn}(u, k) = \frac{1}{(1 + \tan^2 \varphi_0)^{\frac{1}{2}}}.$$

When $\frac{u}{K} > \frac{1}{2}$ a numerically better formula can be obtained if we use (3,1); then

$$\tan \varphi_3^* = \sinh \{a_3(K-u)\}$$

and we get

$$\text{cn}(u, k) \approx \frac{k' \tan \varphi_0^*}{(1 + k'^2 \tan^2 \varphi_0^*)^{\frac{1}{2}}}.$$

In the same way we get

$$\operatorname{dn}(u, k) \approx \left(\frac{1+k'^2 \tan^2 \varphi_0}{1+\tan^2 \varphi_0} \right)^{\frac{1}{2}} \text{ when } \frac{u}{K} \leq \frac{1}{2},$$

$$\operatorname{dn}(u, k) \approx k' \left(\frac{1+\tan^2 \varphi_0^*}{1+k'^2 \tan^2 \varphi_0^*} \right)^{\frac{1}{2}} \text{ when } \frac{u}{K} > \frac{1}{2}.$$

Test values could be checked with a 12 decimal table [8].
Again the relative error was of the order 10^{-12} .

The ALGOL-60 procedures will now be described.

comment procedure for the Complete Elliptic Integral of the first kind
(formula (1,5)), $A = \arcsin(k)$, $0 \leq A \leq \pi/2$;

real procedure K (A); value A; real A;
begin real a1, a2, b, k1; integer n; b := abs (sin (A)); if b > .9539
 then begin a1 := (1+b)/2; b := sqrt(b); a2 := (a1+b)/2;
 b := sqrt (a1xb); K := ln (128x(a2+b)xa2xa1^{1/2}/
 k1^{1/4})/(a2+b)/2
 end
 else begin b := abs (cos (A)); a1 := 1; for n:=1, 2, 3 do
 begin a2 := (a1+b)/2; b := sqrt (a1xb); a1 := a2
 end; K := 3.14159265359/(a1+b)
 end
end K;

comment procedure for the Complete Elliptic Integral of the second kind
(formula (1,6)), $A = \arcsin(k)$, $0 \leq A \leq \pi/2$;

real procedure E (A); value A; real A;
begin real a1, a2, b, s, k1; integer n; b := abs (sin (A));
 k1 := abs (cos (A)); if b > .9539
 then begin a1 := (1+b)/2; s := k1xk1/2+a1xa1-b; b := sqrt (b);
 a2 := (a1+b)/2; A := a1xb; s := s/2+a2xa2-A;
 b := sqrt (A); E := (a2+b)/2+(s/(a2+b))x ln (128x
 (a2+b)xa2xa1xa1/(k1^{1/4}))
 end
 else begin b := k1; a1 := 1; s := 1 + b x b;
 for n:= 1, 2, 3 do
 begin a2 := (a1+b)/2; A := a1xb; a1 := a2;
 s := s/2-a1xa1+A; b := sqrt (A)
 end; E := 12.5663706144 x s /(a1+b)
 end
end E;

comment procedure for the Complete Elliptic Integral (formula (1,7)),
 $A = \arcsin(k)$, $0 \leq A \leq \pi/2$;

real procedure B (A); value A; real A;
begin real a1, a2, b, b1, s, k1; integer n; b := abs (sin (A));
 k1 := abs (cos (A)); b1 := b; if b > .9539
 then begin a1 := (1+b)/2; s := a1xa1 - k1xk1 /2 -b;
 b := sqrt (b); a2 := (a1+b)/2; A := a1xb;
 b := sqrt (A); s := s/2+a2xa2-A;
 B := ((a2+b)/2+(s/(a2+b))x ln (128x(a2+b)xa2xa1x
 a1/(k1^{1/4})))/(b1xb1)
 end
 else begin b := k1; a1 := 1; s := 0; k1 := 1;
 for n:= 1, 2, 3 do
 begin k1 := .25 x k1 x (a1 - b)/(a1 + b);
 a2 := (a1+b)/2; A := a1xb; a1 := a2;
 s := s/2 - k1; b := sqrt (A)
 end; B := 3.14159265359 x (.5 + s x 4)/(a1 + b)
 end
end B;


```

comment procedure for the Incomplete Elliptic Integral of the first kind
(formula (1,1)), A = arcsin (k), P =  $\varphi$ ,  $0 \leq A$ ,  $P \leq \pi/2$ ;
real procedure F (A, P); value A, P; real A, P;
begin real a, al, b, bl, si; integer n, m; b:= abs (cos (A));
  bl:= abs (sin (A)); if bl < .9539
  then begin a:= 1; si:= cos (P)/sin (P); n:=0; A:= b;
    for m:= 1, 2, 3 do
      begin si:= (si-A/si)/2; n:= 2*n + (1 - sign (si))
        /2; a:= (a+b)/2; b:= sqrt (A); A:= a*b
      end; si:= (si - A/si)/2; n:= 2*n + 1 - sign (si);
      a:= (a+b)/2; F:= ((2 * arctan ( a/si) + n *
        3.14159265359)/a)/32
    end
  else begin a:= 1; b:= bl; si:= cos (P)/sin (P); al:= b*b;
    for m:= 1, 2 do
      begin A:= a*b; si:= a*si + sqrt (a*a*(si*si + 1)
        - al); a:= (a+b)/2; si:= si/(a*2);
        b:= sqrt (A); al:= A
      end; si:= a*si + sqrt (a*a*(si*si + 1) - A);
      a:= (a+b)/2; si:= si/(a*2); F:= ln ((1 + sqrt
        (1 + si*si))/si)/a
    end
  end F;

comment procedure for the Incomplete Elliptic Integral of the second
kind (formula (1,2)), A = arcsin (k), P =  $\varphi$ ,  $0 \leq A$ ,  $P \leq \pi/2$ ;
real procedure E (A, P); value A, P; real A, P;
begin real U, V;
  procedure EA;
  begin real a, al, b, S1, S2, si, co; integer n, m;
    b:= U; a:= 1; co:= cos(P)/sin (P); S1:= S2:= 0; al:= b*b;
    n:= 0; for m:= 1, 2, 3, 4 do
      begin A:= a*b; co:= (co - A/co)/2; n:= 2*n + (1 - sign
        (co))/2; P:= a*a; a:= (a+b)/2; P:= P - al;
        S2:= S2 + P * sign (1.5 - n + (n-4)*4)/sqrt
          a*a + co*co; S1:= S1/2 + P; al:= A; b:= sqrt (A)
      end; E:= (((2 * arctan (a/co) + (n + (1 - sign (co))/2) *
        3.14159265359) * (.250 - S1)/a)/2 + S2)/4
    end;
  procedure EB;
  begin real a, S1, co, si; real array b [0:3], SIN [0:3];
    integer m; a:= 1; S1:= U * U; b [0]:= V; si:= sin (P);
    co:= cos (P)/si; SIN [0]:= si; b [3]:= 1 + V;
    co:= (co + sqrt (co * co + 1 - V * V))/b [3]; si:= co*co;
    for m:= 0, 1, 2 do
      begin SIN [m+1]:= 1/sqrt (1 + si); A:= a * b[m];
        a:= b [3]/2; b [m+1]:= sqrt (A); P:= a * a;
        S1:= S1/2 + P - A; b [3]:= a + b [m+1];
        co:= (a * co + sqrt (P * (si + 1) - A))/b [3];
        si:= co * co
      end; si:= sqrt (1 + si); P:= 1/si; U:= 0; b [3]:= a;
      for m:= 3, 2, 1, 0 do U:= 2 * U + b [m] * (P - SIN [m]);
      E:= 4 * S1 * ln ((1 + si)/co)/a + P + U
    end;
  U:= abs (cos (A)); V:= abs (sin (A)); if V < .9539 then EA else EB
end E;

```


comment procedure for the Incomplete Elliptic Integral (formula (1,3)),
 $A = \arcsin(k)$, $P = \varphi$, $0 \leq A$, $P \leq \pi/2$;

real procedure B (A, P); value A, P; real A, P;

begin real U, V;

procedure BA;

begin real a, b, S1, S2, co; integer n, m; array p [0:4];
 b := U; a := 1; co := cos(P)/sin(P); S2 := S1 := 0; n := 0;
 p [0] := 1; for m := 1, 2, 3, 4 do
begin A := a × b; co := (co - A/co)/2; n := 2 × n + (1 - sign
 (co))/2; p [m] := .25 × p [m - 1] × (a - b)/(a + b);
 a := (a + b)/2; S1 := S1/2 + p [m]; S2 := S2 + p [m - 1]
 × sign (1.5 - n + (n:4) × 4)/sqrt (a × a + co × co);
 b := sqrt (A)
end; B := (2 × arctan (a/co) + (n + (1 - sign (co))/2) ×
 3.14159265359) × (.015625 - S1/4)/a + S2/4

end;

procedure BB;

begin real a, S1, co, si; real array b [0:3], SIN [0:3];
integer m; a := 1; S1 := - U × U; b [0] := V; si := sin (P);
 co := cos (P)/si; SIN [0] := si; b [3] := 1 + V; V := V × V;
 co := (co + sqrt (co × co + 1 - V))/b [3]; si := co × co;
for m := 0, 1, 2 do
begin SIN [m+1] := 1/sqrt (1 + si); A := a × b [m];
 a := b [3]/2; b [m+1] := sqrt (A); P := a × a;
 S1 := S1/2 + P - A; b [3] := a + b [m+1]; co := (a × co
 + sqrt (P × (si + 1) - A))/b [3]; si := co × co
end; si := sqrt (1 + si); P := 1/si; U := 0; b [3] := a;
for m := 3, 2, 1, 0 do U := 2 × U + b [m] × (P - SIN [m]);
 B := (4 × S1 × ln ((1 + si)/co))/a + P + U)/V

end;

U := abs (cos (A)); V := abs (sin (A)); if V < .9539 then BA else BB

end B;

comment procedure for the Jacobian Elliptic Function (formula (1,8)),
 $A = \arcsin(k)$, $0 \leq A \leq \pi/2$, $0 \leq u \leq K(k)$;

real procedure sn (u, A); value A, u; real A, u;

begin real array a [0:3], b [0:3]; real a1, t; integer i;

a [0] := 1; b [0] := abs (sin (A)); if b [0] ≥ .9539

then begin for i := 1, 2 do

begin a1 := (a [i-1] + b [i-1])/2;

b [i] := sqrt (a [i-1] × b [i-1]); a [i] := a1

end; a1 := exp ((a [2] + b [2]) × u); t := (a1 - 1)/

(2 × sqrt (a1)); for i := 2, 1, 0 do t := 2 × a [i] ×

t/(a [i] + b [i] - (a [i] - b [i]) × t × t);

sn := t/sqrt (1 + t × t)

end

else

begin

b [0] := abs (cos (A)); for i := 1, 2, 3 do

begin a1 := (a [i-1] + b [i-1])/2; b [i] := sqrt (

a [i-1] × b [i-1]); a [i] := a1

end; a1 := (a [3] + b [3]) × u/2; t := sin (a1);

for i := 3, 2, 1, 0 do t := 2 × a [i] × t/(a [i] +

b [i] + (a [i] - b [i]) × t × t); sn := t

end

end sn;


```

comment procedure for the Jacobian Elliptic Function (formula (1,9)),
A = arcsin (k),  $0 < A < \pi/2$ ,  $0 < u < K(k)$ ;
real procedure cn(u, A); value A, u; real A, u;
begin  real array a [0:3], b [0:3]; real a1, b1, t, k1; integer i;
      a [0] := 1; b1 := b [0] := abs (sin (A)); k1 := abs (cos (A));
      if b [0] > .9539 then
        begin  for i := 1, 2 do
              begin  a1 := (a [i-1] + b [i-1])/2; b [i] := sqrt
                    (a [i-1] × b [i-1]); a [i] := a1
              end; a1 := (ln ((a [2] + b [2]) × a [2] × a [1] × a [1])/2
                    + 2.42601513194 - 2 × ln (k1))/(a [2] + b [2]);
              if u/a1 > .5 then
                begin  a1 := exp ((a [2] + b [2]) × (a1 - u));
                      t := (a1 - 1)/(2 × sqrt (a1)); for i := 2, 1, 0 do
                      t := 2 × a [i] × t / (a [i] + b [i] - (a [i] - b [i])
                      × t × t); cn := k1 × t / sqrt (1 + (k1 × t)2)
                end
              end else
                begin  a1 := exp ((a [2] + b [2]) × u); t := (a1 - 1)/(2 ×
                      sqrt (a1)); for i := 2, 1, 0 do
                      t := 2 × a [i] × t / (a [i] + b [i] - (a [i] - b [i])
                      × t × t); cn := 1/sqrt (1 + t × t)
                end
              end else
                begin  b [0] := k1; for i := 1, 2, 3 do
                      begin  a1 := (a [i-1] + b [i-1])/2; b [i] := sqrt
                            (a [i-1] × b [i-1]); a [i] := a1
                      end; a1 := (a [3] + b [3]) × u/2; if a1 < .78539816340 then
                      begin  t := sin (a1); for i := 3, 2, 1, 0 do
                            t := 2 × a [i] × t / (a [i] + b [i] + (a [i] - b [i])
                            × t × t); cn := sqrt (1 - t × t)
                      end
                      end else
                      begin  t := sin (1.57079632679 - a1);
                            for i := 3, 2, 1, 0 do t := 2 × a [i] × t / (a [i] +
                            b [i] + (a [i] - b [i]) × t × t);
                            cn := b [0] × t / sqrt (1 - (b1 × t)2)
                      end
                end
              end
end cn;
end

```



```

comment procedure for the Jacobian Elliptic Function (formula (1,10)),
A = arcsin (k),  $0 \leq A \leq \pi/2$ ,  $0 \leq u \leq K(k)$ ;
real procedure dn (u, A); value A, u; real A, u;
begin  real array a [0:3], b [0:3]; real a1, b1, t, k1; integer i;
      a [0] := 1; b [0] := b1 := abs (sin (A)); k1 := abs (cos (A));
      if b [0] > .9539 then
        begin  for i := 1, 2 do
              begin  a1 := (a [i-1] + b [i-1])/2; b [i] := sqrt
                    (a [i-1] × b [i-1]); a [i] := a1
              end; a1 := (ln ((a [2] + b [2]) × a [2] × a [1] × a [1])/2
                    + 2.42601513194 - 2 × ln (k1))/(a [2] + b [2]);
              if u/a1 > .5 then
                begin  a1 := exp ((a [2] + b [2]) × (a1 - u));
                      t := (a1 - 1)/(2 × sqrt (a1)); for i := 2, 1, 0 do
                      t := 2 × a [i] × t / (a [i] + b [i] - (a [i] - b [i])
                      × t × t); dn := k1 × sqrt ((1 + t × t)/
                      (1 + (k1 × t)2))
                end else
                begin  a1 := exp ((a [2] + b [2]) × u); t := (a1 - 1)/
                      (2 × sqrt (a1)); for i := 2, 1, 0 do
                      t := 2 × a [i] × t / (a [i] + b [i] - (a [i] - b [i])
                      × t × t); dn := sqrt ((1 + (k1 × t)2)/(1 + t × t))
                end
              end else
            begin  b [0] := k1; for i := 1, 2, 3 do
                  begin  a1 := (a [i-1] + b [i-1])/2; b [i] := sqrt
                        (a [i-1] × b [i-1]); a [i] := a1
                  end; a1 := (a [3] + b [3]) × u/2;
                  if a1 < .78539816340 then
                    begin  t := sin (a1); for i := 3, 2, 1, 0 do
                          t := 2 × a [i] × t / (a [i] + b [i] + (a [i] - b [i])
                          × t × t); dn := sqrt (1 - (b1 × t)2)
                    end else
                    begin  t := sin (1.57079632679 - a1); for i := 3, 2, 1, 0 do
                          t := 2 × a [i] × t / (a [i] + b [i] + (a [i] - b [i])
                          × t × t); dn := b [0]/sqrt (1 - (b1 × t)2)
                    end
                  end
            end
          end dn;
end

```


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